Problem 2.12

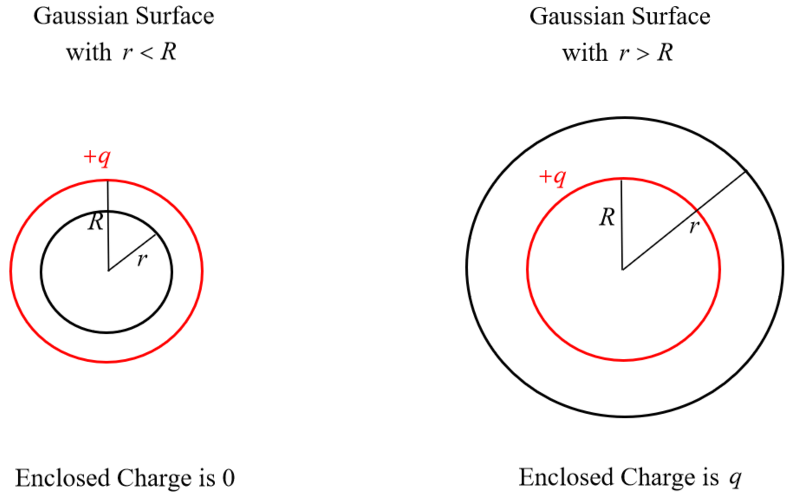
Use Gauss's law to find the electric field inside and outside a spherical shell of radius R that carries a uniform surface charge density σ. Compare your answer to Prob. 2.7; notice how much quicker and easier Gauss's method is.

Solution

One of the governing equations for the electric field in vacuum is Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Normally the curl of $\mathbf{E}$ is also necessary to determine $\mathbf{E}$, but because of the spherical symmetry, the divergence is sufficient. Integrate both sides over the volume of a (black) concentric spherical Gaussian surface with radius r. Two cases need to be considered: (1) $r < R$ and (2) $r > R$.



$$\iiint\limits_{x_0^2+y_0^2+z_0^2 \le r^2} \nabla \cdot \mathbf{E}\, dV_0 = \iiint\limits_{x_0^2+y_0^2+z_0^2 \le r^2} \frac{\rho}{\epsilon_0}\, dV_0$$

$$= \frac{1}{\epsilon_0} \iiint\limits_{x_0^2+y_0^2+z_0^2 \le r^2} \rho\, dV_0$$

$$= \begin{cases} \dfrac{1}{\epsilon_0}(0) & \text{if } r < R \\ \dfrac{1}{\epsilon_0}(q) & \text{if } r > R \end{cases}$$

Apply the divergence theorem on the left side.

$$\oiint\limits_{x_0^2+y_0^2+z_0^2 = r^2} \mathbf{E} \cdot d\mathbf{S}_0 = \begin{cases} 0 & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Because of the spherical symmetry, the electric field is entirely radial: $\mathbf{E} = E(r)\hat{\mathbf{r}}$. Note also that the direction of $d\mathbf{S}$ is the outward unit vector to the Gaussian surface.

$$\oiint_{r_0^2=r^2} [E(r_0)\hat{\mathbf{r}}_0] \cdot (\hat{\mathbf{r}}_0\, dS_0) = \begin{cases} 0 & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Evaluate the dot product.

$$\oiint_{r_0=r} E(r)\, dS_0 = \begin{cases} 0 & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

$E(r)$ is constant on the spherical Gaussian surface, so it can be pulled in front of the integral.

$$E(r) \oiint_{r_0=r} dS = \begin{cases} 0 & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Evaluate the surface integral.

$$E(r)(4\pi r^2) = \begin{cases} 0 & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Solve for $E(r)$.

$$E(r) = \begin{cases} 0 & \text{if } r < R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2} & \text{if } r > R \end{cases}$$

Therefore, the electric field around the spherical shell is

$$\mathbf{E} = \begin{cases} \mathbf{0} & \text{if } r < R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} & \text{if } r > R \end{cases}.$$

This is the same result obtained in Problem 2.7 but with r instead of z. In terms of the given surface charge density $\sigma = q/(4\pi R^2)$, it is

$$\mathbf{E} = \begin{cases} 0 & \text{if } r < R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{\sigma(4\pi R^2)}{r^2}\hat{\mathbf{r}} & \text{if } r > R \end{cases} = \begin{cases} \mathbf{0} & \text{if } r < R \\ \dfrac{\sigma}{\epsilon_0}\dfrac{R^2}{r^2}\hat{\mathbf{r}} & \text{if } r > R \end{cases}.$$

Notice that in the limit as $r \to R^+$, the electric field magnitude is σ/ϵ_0, the same as for the infinite plane.